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THE ROLE OF COUNTABLE PARACOMPACTNESS FOR CONTINUOUS SELECTIONS AVOIDING EXTREME POINTS

By

Takamitsu Yamauchi

Abstract. The role of countable paracompactness to obtain a (set-valued) selection avoiding extreme points is investigated. In particular, we prove the following: Let X be a topological space, Y a normed space and φ a lower semicontinuous compact- and convex-valued mapping of X to Y. If one of the following condition is valid, then φ admits a lower semicontinuous set-valued selection ϕ such that $\phi(x)$ is compact and convex, and each point of $\phi(x)$ is not an extreme point of $\varphi(x)$ for each $x \in X$; (1) the infimum of the set of all diameters of $\varphi(x)$ with $x \in X$ is positive, (2) X is countably paracompact and the cardinality of $\varphi(x)$ is more than one for each $x \in X$. We also give characterizations of some topological spaces in terms of (set-valued) selections avoiding extreme points.

1. Introduction

Throughout this paper, spaces are assumed to be T_1-spaces and λ stands for an infinite cardinal number. Let X and Y be spaces and 2^Y the set of all non-empty subsets of Y. For a mapping $\varphi : X \to 2^Y$, a mapping $f : X \to Y$ is called a *selection* of φ if $f(x) \in \varphi(x)$ for each $x \in X$. Since E. Michael's paper [14], various continuous selection theorems have been obtained (see [20], [21]). Theorem 1.1 below due to E. Michael [14] and M. M. Čoban and V. Valov [2] is fundamental in our study. Before stating Theorem 1.1, let us recall terminology. Let $\mathscr{C}_c(Y)$ denote the set of all compact convex subsets of a normed space Y, and

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the set $\mathscr{C}_c(Y) \cup \{Y\}$ is denoted as $\mathscr{C}'_c(Y)$. A mapping $\varphi : X \to 2^Y$ is called *lower semicontinuous* (*l.s.c.* for short) if the set $\varphi^{-1}[V] = \{x \in X \mid \varphi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. Let Card A denote the cardinality of a set A. A T_1-space X is called *λ-collectionwise normal* if for every discrete collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of X with Card $A \leq \lambda$, there exists a disjoint collection $\{U_\alpha \mid \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$. The weight of a space Y is denoted by $w(Y)$.

THEOREM 1.1 ([14, Theorem 3.2′], [2, Theorem 1], see also [18, Theorem 4.2]). *A T_1-space X is λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}'_c(Y)$ admits a continuous selection.*

Theorem 1.1 is an extension of Katětov-Tong's insertion theorem for normal spaces ([9], [10], [25]). On the other hand, V. Gutev, H. Ohta and K. Yamazaki [5] extended Dowker-Katětov's insertion theorem for countably paracompact normal spaces ([3], [9]) to a theorem on continuous selections avoiding extreme points. For a closed convex subset K of a normed space Y, a point $y \in K$ is called an *extreme point* if every open line segment containing y is not contained in K. For a closed convex subset K of Y, the *weak convex interior* wci(K) of K ([5]) is the set of all non-extreme points of K, that is,

$$\mathrm{wci}(K) = \{y \in K \mid y = \delta y_1 + (1-\delta)y_2 \text{ for some } y_1, y_2 \in K \setminus \{y\} \text{ and } 0 < \delta < 1\}.$$

A space X is *countably paracompact* if every countable open cover is refined by a locally finite open cover of X. V. Gutev, H. Ohta and K. Yamazaki [5] established the following theorem for generalized $c_0(\lambda)$-space Y, and the author [26] extended their theorem to a Banach space Y of weight $\leq \lambda$.

THEOREM 1.2 ([5, Theorems 4.5], [26, Theorem 2]). *A T_1-space X is countably paracompact and λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}'_c(Y)$ with* Card $\varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Although the existence itself of a continuous selection is guaranteed by Theorem 1.1, the assumption in Theorem 1.2 that X is countably paracompact can not be dropped. Suggested by Theorem 1.2, we are concerned with the role of countable paracompactness to obtain a continuous selections avoiding extreme points. Our study has two directions.

The first one is to show that if the diameters of the values of an l.s.c. mapping $\varphi : X \to \mathscr{C}_c'(Y)$ is uniformly large, then the countable paracompactness of X is not necessary. For a subset A of a metric space (Y, d), let $\operatorname{diam} A = \sup\{d(y_1, y_2) \mid y_1, y_2 \in A\}$. If Y is a normed space, a metric on Y is assumed to be induced by the norm of Y.

Theorem 1.3. *A T_1-space X is λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \le \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c'(Y)$ with $\inf\{\operatorname{diam} \varphi(x) \mid x \in X\} > 0$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

For a mapping $\varphi : X \to 2^Y$, a mapping $\phi : X \to 2^Y$ is called a *set-valued selection* if $\phi(x) \subset \varphi(x)$ for each $x \in X$. For compact- and convex-valued l.s.c. mappings, we prove the following.

Proposition 1.4. *Let X be a topological space, Y a normed space and $\varphi : X \to \mathscr{C}_c(Y)$ an l.s.c. mapping such that $\inf\{\operatorname{diam} \varphi(x) \mid x \in X\} > 0$. Then φ admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(Y)$ such that $\phi(x) \subset \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

The second direction is to show that the countable metacompactness (rather than countable paracompactness) of X is necessary in order that every l.s.c. mapping $\varphi : X \to \mathscr{C}_c(\mathbf{R})$ with $\operatorname{Card} \varphi(x) > 1$ for each $x \in X$ may admit an l.s.c. compact- and convex-valued set-valued selection avoiding extreme points, where $\mathbf{R}$ is the space of all real numbers with the usual topology. A topological space X is called *countably metacompact* if every countable open cover of X is refined by a point-finite open cover of X. Note that every countably paracompact space is countably metacompact, and every countably metacompact normal space is countably paracompact (see [7]).

Theorem 1.5. *A topological space X is countably metacompact if and only if for every normed space Y, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c(Y)$ with $\operatorname{Card} \varphi(x) > 1$ for each $x \in X$ admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(Y)$ such that $\phi(x) \subset \operatorname{wci}(\varphi(x))$ for each $x \in X$. In this statement, "every normed space Y" can be replaced with "$Y = \mathbf{R}$".*

The statement in the abstract follows from Proposition 1.4 and Theorem 1.5. In the realm of normal spaces, an l.s.c. mapping $\varphi : X \to \mathscr{C}_c(Y)$ in Theorem 1.5

can be replaced with an l.s.c. mapping $\varphi : X \to \mathscr{C}_c'(Y)$ if the domain space X is assumed to be almost expandable. A space X is *almost λ-expandable* ([11], [24]) if for every locally finite collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of X with $\mathrm{Card}\, A \le \lambda$, there exists a point-finite collection $\{U_\alpha \mid \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$. Every almost λ-expandable space is countably metacompact ([11, Theorem 2.6]). We prove the following.

THEOREM 1.6. *A normal space X is almost λ-expandable if and only if for every Banach space Y with $w(Y) \le \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c'(Y)$ with $\mathrm{Card}\, \varphi(x) > 1$ for each $x \in X$ admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(Y)$ such that $\phi(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Some preliminary results are shown in section 2. In section 3, some results on l.s.c. set-valued selections are obtained. In particular, Proposition 1.4 and Theorems 1.5 and 1.6 are proved. In section 4, we prove Theorem 1.3 and some results on continuous selections avoiding extreme points.

2. Preliminaries

Let $\mathbf{N}$ denote the set of all positive integers. The closure of a subset S of a topological space is denoted by $\mathrm{Cl}\, S$. A subset A of a topological space X is called a *cozero-set* if there is a continuous function $f : X \to \mathbf{R}$ such that $A = \{x \in X \mid f(x) \neq 0\}$. Let (Y, d) be a metric space. For $A, B \in 2^Y$, let $\mathrm{dist}(A, B) = \inf\{d(y_1, y_2) \mid y_1 \in A, y_2 \in B\}$. For $y \in Y$, $A \subset Y$ and $\varepsilon > 0$, let $B(y, \varepsilon) = \{z \in Y \mid d(y, z) < \varepsilon\}$ and $B(A, \varepsilon) = \bigcup_{y \in A} B(y, \varepsilon)$. For a collection $\mathscr{V}$ of subsets of Y, put $\mathrm{mesh}\, \mathscr{V} = \sup\{\mathrm{diam}\, V \mid V \in \mathscr{V}\}$. By $\mathscr{F}(Y)$ (respectively, $\mathscr{C}(Y)$) we denote the set of all non-empty closed (respectively, compact) subsets of Y, and put $\mathscr{C}'(Y) = \mathscr{C}(Y) \cup \{Y\}$. Let $\mathscr{F}_c(Y)$ denote the set of all non-empty closed convex subsets of a normed space Y. We denote by $\mathrm{conv}\, A$ the convex hull of a subset A of a normed space Y. For other undefined terminology, we refer to [4].

Let us recall that a key to obtain a continuous selection avoiding extreme points is to construct two continuous selections with different values locally (see the proof of (1) $\Rightarrow$ (2) of [5, Theorem 4.5]).

LEMMA 2.1 ([5]). *Let X be a topological space, Y a normed space and $\varphi : X \to \mathscr{F}_c(Y)$ a mapping. Suppose that there exist a locally finite cozero-set cover $\mathscr{U}$ of X and a collection $\{f_U \mid U \in \mathscr{U}\}$ of continuous mappings $f_U : U \to Y$, $U \in \mathscr{U}$, such that $f_U(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in U$. Then φ admits a continuous selection $f : X \to Y$ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

In the above, to obtain continuous mappings $f_U : U \to Y$, $U \in \mathscr{U}$, *such that* $f_U(x) \in \mathrm{wci}(\varphi(x))$ *for each* $x \in U$, *it suffices to find continuous selections* $f_U^1, f_U^2 : U \to Y$ *of* $\varphi|_U$ *such that* $f_U^1(x) \neq f_U^2(x)$ *for each* $x \in U$.

The following lemma is useful to obtain an l.s.c. set-valued selections avoiding extreme points.

Lemma 2.2. *Let* X *be a topological space,* Y *a normed space and* $\varphi : X \to \mathscr{F}_c(Y)$ *a mapping. Suppose that there exist a point-finite open cover* $\mathscr{U}$ *of* X *and a collection* $\{\phi_U \mid U \in \mathscr{U}\}$ *of l.s.c. mappings* $\phi_U : U \to \mathscr{C}_c(Y)$, $U \in \mathscr{U}$, *such that* $\phi_U(x) \subset \mathrm{wci}(\varphi(x))$ *for each* $x \in U$. *Then* φ *admits an l.s.c. set-valued selection* $\phi : X \to \mathscr{C}_c(Y)$ *such that* $\phi(x) \subset \mathrm{wci}(\varphi(x))$ *for each* $x \in X$.

In the above, to obtain l.s.c. mappings $\phi_U : U \to \mathscr{C}_c(Y)$, $U \in \mathscr{U}$, *such that* $\phi_U(x) \subset \mathrm{wci}(\varphi(x))$ *for each* $x \in U$, *it suffices to find l.s.c. set-valued selections* $\phi_U^1, \phi_U^2 : U \to \mathscr{C}_c(Y)$ *of* $\varphi|_U$ *such that* $\phi_U^1(x) \cap \phi_U^2(x) = \varnothing$ *for each* $x \in U$.

Proof. Assume that $\phi_U^1, \phi_U^2 : U \to \mathscr{C}_c(Y)$ are l.s.c. set-valued selections of $\varphi|_U$ such that $\phi_U^1(x) \cap \phi_U^2(x) = \varnothing$ for each $x \in U$. Define a mapping $\phi_U : U \to 2^Y$ by $\phi_U(x) = \{(y_1 + y_2)/2 \mid y_1 \in \phi_U^1(x), y_2 \in \phi_U^2(x)\}$ for $x \in U$. Then ϕ_U is l.s.c. (cf. [6, Proposition 2.59]) and $\phi_U(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in U$. Since $\phi_U^1(x)$, $\phi_U^2(x) \in \mathscr{C}_c(Y)$, we have $\phi_U(x) \in \mathscr{C}_c(Y)$ for each $x \in X$. Define a mapping $\phi : X \to 2^Y$ by $\phi(x) = \mathrm{conv} \bigcup \{\phi_U(x) \mid U \in \mathscr{U} \text{ with } x \in U\}$ for each $x \in U$. Then $\phi(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in X$. By using [14, Proposition 2.1], it is easy to see that ϕ is l.s.c. Since $\mathscr{U}$ is point-finite, each $\phi(x)$ is compact (cf. [12, Lemma 2.10.14]). □

A space X is called *almost discretely λ-expandable* ([24]) if for every discrete collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of X with $\mathrm{Card}\, A \leq \lambda$, there exists a point-finite collection $\{U_\alpha \mid \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$. Note that a space is almost λ-expandable if and only if it is countably metacompact and almost discretely λ-expandable ([24, Theorem 2.8]). J. C. Smith [23, Theorem 2.7] proved that a space X is almost discretely λ-expandable if and only if for every closed subset F of X and for every open cover $\mathscr{V}$ of F of finite order, there exists a point-finite open cover $\mathscr{U}$ of X such that $\{U \cap F \mid U \in \mathscr{U}\}$ refines $\mathscr{V}$. In this statement, "open cover $\mathscr{V}$ of F of finite order" can be replaced by "point-finite open cover $\mathscr{V}$ of F" for a normal space X. This is probably known, but for the sake of completeness we give a proof.

Proposition 2.3. *A normal space X is almost discretely λ-expandable if and only if for every closed subset F of X and for every point-finite open cover $\mathscr{V}$ of F with* Card $\mathscr{V} \leq \lambda$, *there exists a point-finite open cover $\mathscr{U}$ of X such that $\{U \cap F \mid U \in \mathscr{U}\}$ refines $\mathscr{V}$.*

Proof. It suffices to show the "only if" part. Let X be an almost discretely λ-expandable space, F a closed subset of X and $\mathscr{V}$ a point-finite open cover of F with Card $\mathscr{V} \leq \lambda$. For each $i \in \mathbf{N}$, put $F_i = \{x \in X \mid \mathrm{Card}\{V \in \mathscr{V} \mid x \in V\} \leq i\}$. By an argument similar to [4, Theorem 5.3.3], we can take a sequence $\{W_i \mid i \in \mathbf{N}\}$ of open subsets of X and a point-finite collections $\mathscr{U}_i$, $i \in \mathbf{N}$, of open subsets of X such that $F_i \subset \bigcup_{j=1}^{i} W_j$, $F_i \backslash \bigcup_{j=1}^{i-1} W_j \subset W_i \subset \mathrm{Cl}\, W_i \subset \bigcup \mathscr{U}_i$ and $\{U \cap F \mid U \in \mathscr{U}_i\}$ refines $\mathscr{V}$ for each $i \in \mathbf{N}$. Put $W = \bigcup_{i \in \mathbf{N}} W_i$. Then the collection $\mathscr{U} = \{U \cap (W \backslash \bigcup_{j=1}^{i-1} \mathrm{Cl}\, W_j) \mid U \in \mathscr{U}_i, i \in \mathbf{N}\} \cup \{X \backslash F\}$ is the required one. □

A space X is called *λ-metacompact* if every open cover $\mathscr{U}$ of X with Card $\mathscr{U} \leq \lambda$ is refined by a point-finite open cover of X. M. M. Čoban [1, Theorem 6.1] characterized λ-metacompact spaces in terms of l.s.c. set-valued selections. By an argument similar to [1, Theorem 6.1] (see also [18], [20, A §4]) with Proposition 2.3, we have the following.

Proposition 2.4. *A normal space X is almost discretely λ-expandable if and only if for every completely metrizable space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}'(Y)$ admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}(Y)$.*

Lemma 2.5. *Let X be an almost discretely λ-expandable normal space* (*respectively, a λ-metacompact regular space*), *(Y, d) a complete metric space with $w(Y) \leq \lambda$ and $\varphi : X \to \mathscr{C}'(Y)$* (*respectively, $\varphi : X \to \mathscr{F}(Y)$*) *an l.s.c. mapping such that* $\inf\{\mathrm{diam}\, \varphi(x) \mid x \in X\} > 0$. *Then φ admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}(Y)$ such that* $\inf\{\mathrm{diam}\, \phi(x) \mid x \in X\} > 0$.

Proof. Let $c = \inf\{\mathrm{diam}\, \varphi(x) \mid x \in X\}$. Take a locally finite open cover $\mathscr{V}$ of Y such that mesh $\mathscr{V} < c/4$. Put $\mathscr{W} = \{\varphi^{-1}[V^1] \cap \varphi^{-1}[V^2] \mid V^1, V^2 \in \mathscr{V}$, $\mathrm{dist}(V^1, V^2) > c/4\}$. Then $\mathscr{W}$ is an open cover of X with Card $\mathscr{W} \leq \lambda$, and there exists a point-finite open cover $\mathscr{U}$ of X such that $\{\mathrm{Cl}\, U \mid U \in \mathscr{U}\}$ refines $\mathscr{W}$. (If X is almost discretely λ-expandable and normal, fix non-empty $V_0^1, V_0^2 \in \mathscr{V}$ such that $\mathrm{dist}(V_0^1, V_0^2) > c/4$. Then $F = X \backslash (\varphi^{-1}[V_0^1] \cap \varphi^{-1}[V_0^2])$ is closed in X, $\varphi(x)$ is compact for each $x \in F$, and $\mathscr{W}$ is point-finite at each point of F. Thus the existence of such a cover $\mathscr{U}$ follows from Proposition 2.3 and [4, Theorem 1.5.18].) For each $U \in \mathscr{U}$, take $V_U^1, V_U^2 \in \mathscr{V}$ such that $\mathrm{Cl}\, U \subset \varphi^{-1}[V_U^1] \cap \varphi^{-1}[V_U^2]$ and $\mathrm{dist}(V_U^1, V_U^2) > c/4$. For $i = 1, 2$, define a mapping $\varphi_U^i : \mathrm{Cl}\, U \to \mathscr{C}'(\mathrm{Cl}\, V_U^i)$ (respectively, $\varphi_U^i : \mathrm{Cl}\, U \to \mathscr{F}(\mathrm{Cl}\, V_U^i)$) by $\varphi_U^i(x) = \mathrm{Cl}(\varphi(x) \cap V_U^i)$ for each $x \in \mathrm{Cl}\, U$. By Proposition 2.4 (respectively, [1, Theorem 6.1]), there exists an l.s.c. set-valued selection $\phi_U^i : \mathrm{Cl}\, U \to \mathscr{C}(\mathrm{Cl}\, V_U^i)$ of φ_U^i. Then $\mathrm{dist}(\phi_U^1(x), \phi_U^2(x)) > c/4$ for each $x \in \mathrm{Cl}\, U$ and $U \in \mathscr{U}$. Define $\phi : X \to 2^Y$ by $\phi(x) = \bigcup\{\phi_U^i(x) \mid x \in U, i = 1, 2\}$. Then ϕ is the required mapping. □

3. Lower Semicontinuous Set-valued Selections Avoiding Extreme Points

First, we show the following proposition including Proposition 1.4.

Proposition 3.1. *Let X be a topological space (respectively, an almost discretely λ-expandable normal space, a λ-metacompact regular space), Y a normed space (respectively, a Banach space with $w(Y) \le \lambda$, a Banach space with $w(Y) \le \lambda$) and $\varphi : X \to \mathscr{C}_c(Y)$ (respectively, $\varphi : X \to \mathscr{C}_c'(Y)$, $\varphi : X \to \mathscr{F}_c(Y)$) an l.s.c. mapping such that $\inf\{\mathrm{diam}\, \varphi(x) \mid x \in X\} > 0$. Then φ admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(Y)$ such that $\phi(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Proof. Note that if a mapping $\phi : X \to \mathscr{C}(Y)$ is l.s.c., then the mapping $\theta : X \to 2^Y$ defined by $\theta(x) = \mathrm{Cl}(\mathrm{conv}\, \phi(x))$ is l.s.c. ([14, Propositions 2.3 and 2.6]) and compact- and convex-valued (cf. [12, Theorem 2.8.15]). Also note that for each closed convex subsets C and D of Y with $C \subset D$, we have $\mathrm{wci}(C) \subset \mathrm{wci}(D)$. Thus, by Lemma 2.5, we may assume that φ is compact- and convex-valued in each case. Let $c = \inf\{\mathrm{diam}\, \varphi(x) \mid x \in X\}$. Take a locally finite cover $\mathscr{V}$ of Y consisting of non-empty open subsets of Y such that mesh $\mathscr{V} < c/4$. Since each $\varphi(x)$ is compact, the collection $\mathscr{U} = \{\varphi^{-1}[V^1] \cap \varphi^{-1}[V^2] \mid V^1, V^2 \in \mathscr{V}$ with $\mathrm{dist}(\mathrm{conv}\, V^1, \mathrm{conv}\, V^2) > c/4\}$ is a point-finite open cover of X. For each $U \in \mathscr{U}$, there are $V_U^1, V_U^2 \in \mathscr{V}$ such that $\mathrm{dist}(\mathrm{conv}\, V_U^1, \mathrm{conv}\, V_U^2) > c/4$ and $U = \varphi^{-1}[V_U^1] \cap \varphi^{-1}[V_U^2]$. For $i = 1, 2$, define a mapping $\phi_U^i : U \to 2^Y$ by $\phi_U^i(x) = \mathrm{Cl}(\varphi(x) \cap \mathrm{conv}\, V_U^i)$ for each $x \in U$. Then ϕ_U^1 and ϕ_U^2 are set-valued selections of φ such that $\phi_U^1(x), \phi_U^2(x) \in \mathscr{C}_c(Y)$ and $\phi_U^1(x) \cap \phi_U^2(x) = \varnothing$ for each $x \in X$. By [14, Propositions 2.3 and 2.4], ϕ_U^1 and ϕ_U^2 are l.s.c. Thus the conclusion follows from Lemma 2.2. □

Let us prove Theorems 1.5 and 1.6.

LEMMA 3.2. *Let X be a countably metacompact space* (*respectively*, *an almost λ-expandable normal space*, *a λ-metacompact regular space*), *Y a normed space* (*respectively*, *a Banach space with $w(Y) \leq \lambda$*, *a Banach space with $w(Y) \leq \lambda$*) *and $\varphi : X \to \mathscr{C}_c(Y)$* (*respectively*, *$\varphi : X \to \mathscr{C}'_c(Y)$*, *$\varphi : X \to \mathscr{F}_c(Y)$*) *an l.s.c. mapping such that* $\mathrm{Card}\,\varphi(x) > 1$ *for each $x \in X$. Then there exists an l.s.c. mapping $\phi : X \to \mathscr{C}_c(Y)$ such that $\phi(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

PROOF. For each $i \in \mathbf{N}$, put $U_i = \{x \in X \mid \mathrm{diam}\,\varphi(x) > 1/2^i\}$. Then $\{U_i \mid i \in \mathbf{N}\}$ is an open cover of X. Since X is countably metacompact (respectively, almost λ-expandable and normal, λ-metacompact and regular), there is a point-finite open cover $\{V_i \mid i \in \mathbf{N}\}$ of X such that $V_i \subset U_i$ (respectively, $\mathrm{Cl}\,V_i \subset U_i$, $\mathrm{Cl}\,V_i \subset U_i$) for each $i \in \mathbf{N}$. By Proposition 3.1, for each $i \in \mathbf{N}$ there exists an l.s.c. mapping $\phi_i : V_i \to \mathscr{C}_c(Y)$ such that $\phi_i(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in V_i$. Thus the conclusion follows from Lemma 2.2. $\square$

For a set A, let $l_1(A)$ be the Banach space of all functions $s : A \to \mathbf{R}$ such that $\sum_{\alpha \in A} |s(\alpha)| < \infty$, where the linear operations are defined pointwise and $\|s\| = \sum_{\alpha \in A} |s(\alpha)|$ for each $s \in l_1(A)$. We have the following characterizations in terms of l.s.c. set-valued selections avoiding extreme points.

THEOREM 3.3. (1) (Theorem 1.5) *A topological space X is countably metacompact if and only if for every normed space Y, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c(Y)$ with* $\mathrm{Card}\,\varphi(x) > 1$ *for each $x \in X$ admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(Y)$ such that $\phi(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in X$. In this statement*, "*every normed space Y*" *can be replaced with* "$Y = \mathbf{R}$".

(2) *A normal space X is almost discretely λ-expandable if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}'_c(Y)$ with* $\inf\{\mathrm{diam}\,\varphi(x) \mid x \in X\} > 0$ *admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(Y)$ such that $\phi(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

(3) (Theorem 1.6) *A normal space X is almost λ-expandable if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}'_c(Y)$ with* $\mathrm{Card}\,\varphi(x) > 1$ *for each $x \in X$ admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(Y)$ such that $\phi(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

(4) *A regular space X is λ-metacompact if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{F}_c(Y)$ with* $\mathrm{Card}\,\varphi(x) > 1$ *for each $x \in X$ admits an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(Y)$ such that $\phi(x) \subset \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Proof. The "only if" part of (2) follows from Proposition 3.1. The "only if" parts of (1), (3) and (4) follow from Lemma 3.2.

To show the "if" part of (1), we will apply [7, Corollary]. Let $\{U_i \mid i \in \mathbf{N}\}$ be an open cover of X such that $U_i \subset U_{i+1}$ for each $i \in \mathbf{N}$. For each $x \in X$, let $i(x) = \min\{i \in \mathbf{N} \mid x \in U_i\}$. Define an l.s.c. mapping $\varphi : X \to \mathscr{C}_c(\mathbf{R})$ by $\varphi(x) = [0, 1/i(x)]$ for each $x \in X$. By the assumption, there exits an l.s.c. mapping $\phi : X \to \mathscr{C}_c(\mathbf{R})$ such that $\phi(x) \subset \mathrm{wci}(\varphi(x)) = (0, 1/i(x))$ for each $x \in X$. Put $F_i = X \backslash \phi^{-1}[(0, 1/i)]$ for each $i \in \mathbf{N}$. Then $\{F_i \mid i \in \mathbf{N}\}$ is a closed cover of X such that $F_i \subset U_i$ for each $i \in \mathbf{N}$. Thus X countably metacompact due to [7, Corollary].

To show the "if" part of (2), let $\{F_\alpha \mid \alpha \in A\}$ be a discrete collection of closed subsets of X such that $\mathrm{Card}\, A \leq \lambda$. Let $\{e_\alpha \mid \alpha \in A\}$ be the set of standard unit vectors in $l_1(A)$, that is, e_α is the point of $l_1(A)$ defined by $e_\alpha(\beta) = 1$ if $\alpha = \beta$ and $e_\alpha(\beta) = 0$ otherwise. Put $C_\alpha = \mathrm{conv}\{e_\alpha, 2e_\alpha\}$ for each $\alpha \in A$, and define a mapping $\varphi : X \to 2^{l_1(A)}$ by $\varphi(x) = C_\alpha$ if $x \in F_\alpha$ and $\alpha \in A$, and $\varphi(x) = l_1(A)$ otherwise. Then φ is l.s.c., $\varphi(x) \in \mathscr{C}_c'(l_1(A))$ for each $x \in X$ and $\inf\{\mathrm{diam}\, \varphi(x) \mid x \in X\} > 0$. By the assumption, there is an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(l_1(A))$ of φ. Note that $\{B(C_\alpha, 1/4) \mid \alpha \in A\}$ is a discrete collection of open subsets of $l_1(A)$. Thus $\{\phi^{-1}[B(C_\alpha, 1/4)] \mid \alpha \in A\}$ is a point-finite collection of open subsets of X such that $F_\alpha \subset \phi^{-1}(B(C_\alpha, 1/4))$ for each $\alpha \in A$. Therefore X is almost discretely λ-expandable.

The "if" part of (3) follows from those of (1) and (2) since a space X is almost λ-expandable if and only if it is countably metacompact and almost discretely λ-expandable ([24, Theorem 2.8]).

Finally, let us show the "if" part of (4). Let $\mathscr{U}$ be an open cover of X with $\mathrm{Card}\, \mathscr{U} \leq \lambda$. Put $A = \mathscr{U} \times \{0, 1\}$. Following [14, Theorem 3.2″], define an l.s.c. mapping $\varphi : X \to \mathscr{F}_c(l_1(A))$ by $\varphi(x) = \{y \in l_1(A) \mid \|y\| = 1, y(\alpha) \geq 0$ for every $\alpha \in A$, and $y((U, i)) = 0$ for every $U \in \mathscr{U}$ with $x \notin U$ and $i \in \{0, 1\}\}$ for each $x \in X$. Then $\mathrm{Card}\, \varphi(x) > 1$ for each $x \in X$. By the assumption, there exists an l.s.c. set-valued selection $\phi : X \to \mathscr{C}_c(l_1(A))$ of φ. Put $Y = l_1(A) \backslash \{\mathbf{0}\}$, where $\mathbf{0}$ is the origin of $l_1(A)$. By the definition of φ, $\phi(x) \subset Y$ for each $x \in X$. For each $\alpha \in A$, put $V_\alpha = \{y \in l_1(A) \mid y(\alpha) \neq 0\}$. Since $\mathscr{V} = \{V_\alpha \mid \alpha \in A\}$ is an open cover of the metric space Y, there exists a locally finite open cover $\mathscr{W}$ of Y which refines $\mathscr{V}$. Then $\{\phi^{-1}[W] \mid W \in \mathscr{W}\}$ is a point-finite open cover of X which refines $\mathscr{U}$. □

The author does not know whether the assumption in (3) and (4) of Theorem 3.3 that X is normal can be dropped. In our proof, the normality of X was essentially used in Propositions 2.3 and 2.4.

4. Continuous Selections Avoiding Extreme Points

A T_1-space X is *λ-PF-normal* if every point-finite open cover $\mathcal{U}$ of X with Card $\mathcal{U} \leq \lambda$ is normal. λ-PF-normal spaces are first investigated by E. Michael [13], and the name "PF-normal" is due to J. C. Smith [22]. Note that every λ-collectionwise normal space is λ-PF-normal ([13, Theorem 2]) and ω-PF-normality coincides with the normality ([15, Corollary of Theorem 5]), where ω is the first infinite cardinal number. T. Kandô [8] and S. Nedev [18] proved the following selection theorem for λ-PF-normal spaces. Note that in the realm of normal spaces, pointwise-paracompactness in [8] coincides with PF-normality, and λ-pointwise-$\aleph_0$-paracompactness in [18] is the same as λ-PF-normality.

THEOREM 4.1 ([8, Theorem IV], [18, Theorem 4.1]). *A T_1-space X is λ-PF-normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c(Y)$ admits a continuous selection.*

REMARK 4.2. The "if" part of Theorem 4.1 is valid even if the l.s.c. mapping "$\varphi : X \to \mathscr{C}_c(Y)$" is replaced by "$\varphi : X \to \mathscr{C}_c(Y)$ with $\inf\{\operatorname{diam} \varphi(x) \mid x \in X\} > 0$". Indeed, for a point-finite open cover $\mathcal{U}$ of X with Card $\mathcal{U} \leq \lambda$, put $A = \mathcal{U} \times \{0, 1\}$. Following [14, Theorem 3.2″], define an l.s.c. mapping $\varphi : X \to \mathscr{F}_c(l_1(A))$ by $\varphi(x) = \{y \in l_1(A) \mid \|y\| = 1, y(\alpha) \geq 0$ for every $\alpha \in A$, and $y((U, i)) = 0$ for every $U \in \mathcal{U}$ with $x \notin U$ and $i \in \{0, 1\}\}$ for each $x \in X$. Then $\inf\{\operatorname{diam} \varphi(x) \mid x \in X\} \geq 1$ and, since $\mathcal{U}$ is point-finite, $\varphi(x)$ is compact for each $x \in X$. By an argument analogous to the proof of [14, Theorem 3.2″], it is shown that there is a partition of unity subordinated by $\mathcal{U}$. Thus $\mathcal{U}$ is normal due to [16, Theorem 2.1].

A space X is *λ-paracompact* if every open cover $\mathcal{U}$ of X with Card $\mathcal{U} \leq \lambda$ is refined by a locally finite open cover of X. Applying Theorem 4.1 and results in section 3, we have the following characterizations.

THEOREM 4.3. (1) *A T_1-space X is λ-PF-normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c(Y)$ with $\inf\{\operatorname{diam} \varphi(x) \mid x \in X\} > 0$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

(2) *A T_1-space X is countably paracompact and λ-PF-normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c(Y)$ with Card $\varphi(x) > 1$ for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$.*

(3) (Theorem 1.3) *A T_1-space X is λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c'(Y)$ with* $\inf\{\operatorname{diam} \varphi(x) \mid x \in X\} > 0$ *admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

(4) ([5, Theorems 4.5], [26, Theorem 2]) *A T_1-space X is countably paracompact and λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c'(Y)$ with* $\operatorname{Card} \varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

(5) ([26, Theorem 8]) *A T_1-space X is λ-paracompact and normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{F}_c(Y)$ with* $\operatorname{Card} \varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

Proof. By virtue of [5, Theorem 3.1], we have the following two facts: A space is λ-collectionwise normal if and only if it is λ-PF-normal and almost discretely λ-expandable; a space is countably paracompact and λ-collectionwise normal if and only if it is λ-PF-normal and almost λ-expandable. A space is λ-paracompact and normal if and only if it is λ-PF-normal and λ-metacompact. Thus statement (1) (respectively, (2), (3), (4), (5)) follows from Proposition 3.1 (respectively, (1) of Theorem 3.3, (2) of Theorem 3.3, (3) of Theorem 3.3, (4) of Theorem 3.3), Theorem 4.1 and Remark 4.2. □

In [5, Theorem 4.6] and [26, Corollaries 10 and 11], characterizations of perfectly normal λ-collectionwise normal spaces and ones of perfectly normal λ-paracompact spaces are obtained. Analogously, we have the following characterizations of perfectly normal λ-PF-normal spaces.

Theorem 4.4. *A T_1-space X is perfectly normal and λ-PF-normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathscr{C}_c(Y)$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ whenever* $\operatorname{Card} \varphi(x) > 1$.

We need preparation to prove Theorem 4.4. λ-PF-normality is not hereditary with respect to closed subsets unlike normality and collectionwise normality (see [5, p. 506]), but it is hereditary to open F_σ-sets.

Proposition 4.5. *Every open F_σ-set of a λ-PF-normal space is λ-PF-normal.*

PROOF. Let G be an open F_σ-set of a λ-PF-normal space X. Using the normality of X, take a sequence $\{V_i \mid i \in \mathbf{N}\}$ of cozero-sets of X such that $G = \bigcup_{i\in\mathbf{N}} \mathrm{Cl}\, V_i$ and $\mathrm{Cl}\, V_i \subset V_{i+1}$ for each $i \in \mathbf{N}$. Let $\mathscr{U}$ be a point-finite open cover of G with $\mathrm{Card}\, \mathscr{U} \le \lambda$. For each $i \in \mathbf{N}$, put $\mathscr{W}_i = \{U \cap V_{i+1} \mid U \in \mathscr{U}\} \cup \{X \backslash \mathrm{Cl}\, V_i\}$. Then $\mathscr{W}_i$ is a point-finite open cover of X such that $\mathrm{Card}\, \mathscr{W}_i \le \lambda$, and hence it is normal. By virtue of [16, Theorem 1.2], there is a locally finite cozero-set cover $\mathscr{W}_i'$ of X which refines $\mathscr{W}_i$. Put $\mathscr{V}_i = \{W \cap V_i \mid W \in \mathscr{W}_i'\}$. Then $\mathscr{V}_i$ is a locally finite collection of cozero-sets in X which refines $\mathscr{U}$. Thus $\bigcup_{i\in\mathbf{N}} \mathscr{V}_i$ is normal by virtue of [17, Theorem 1.2]. □

The following lemma was essentially proved by V. Gutev, H. Ohta and K. Yamazaki [5] (see the proof of (1) $\Rightarrow$ (2) of [5, Theorem 4.6]).

LEMMA 4.6 ([5]). *Let X be a topological space, Y a normed space, $\varphi : X \to \mathscr{F}_c(Y)$ a mapping and C a cozero-set of X such that $C \subset \{x \in X \mid \mathrm{Card}\, \varphi(x) > 1\}$. If φ admits a continuous selection and the restricted mapping $\varphi|_C : C \to \mathscr{F}_c(Y)$ admits a continuous selection $g : C \to Y$ such that $g(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in C$, then there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in C$.*

PROOF OF THEOREM 4.4. To show the "only if" part, let X be a perfectly normal λ-PF-normal space, and Y and $\varphi : X \to \mathscr{C}_c(Y)$ as in the statement of Theorem 4.4. Then the open subset $C = \{x \in X \mid \mathrm{Card}\, \varphi(x) > 1\}$ of X is an F_σ-set, and hence C is countably paracompact and λ-PF-normal by Proposition 4.5. For the restricted mapping $\varphi|_C : C \to \mathscr{C}_c(Y)$, by (2) of Theorem 4.3, there is a continuous selection $g : C \to Y$ such that $g(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$. By virtue of Theorem 4.1, φ admits a continuous selection. Since C is a cozero-set of X, by Lemma 4.6, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in C$. This f is the required selection.

To show the "if" part, let X be a T_1-space satisfying the condition in Theorem 4.4. By virtue of Theorem 4.1, X is λ-PF-normal. By the same argument as in the proof of (4) $\Rightarrow$ (1) of [5, Theorem 4.6], it is proved that X is perfectly normal. □

REMARK 4.7. In [5], V. Gutev, H. Ohta and K. Yamazaki proved sandwich-like properties for mappings into the Banach space $C_0(Y)$ corresponding to (4) and (5) of Theorem 4.3 (see also [19]). Similarly, we can prove sandwich-like properties corresponding to (1)–(3) of Theorem 4.3 and Theorem 4.4.

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Department of Mathematics
Shimane University
Matsue, 690-8504, Japan

Current address:
Department of General Education
Kure College of Technology
Kure, 737-8506, Japan
E-mail address: yamauchi@kure-nct.ac.jp